

Reg. No. : .....

Name : .....

Sixth Semester B.Sc. Degree Examination, March 2021

First Degree Programme under CBCSS

Mathematics

Core Course

MM 1644 : ABSTRACT ALGEBRA – II  
(2015-2017 Admission)

Time : 3 Hours

Max. Marks : 80

SECTION – I

All the first ten questions are compulsory. Each carries 1 mark.

1. Determine whether the map  $\phi: GL(n, \mathbb{R}) \rightarrow \mathbb{R}$  given by  $\phi(A) = tr(A)$  is a homomorphism, where,  $GL(n, \mathbb{R})$  is the multiplicative group of all invertible  $n \times n$  matrices.
2. Find the order of  $(\mathbb{Z}_4 \times \mathbb{Z}_2) / \langle (2, 1) \rangle$
3. Find the solution of the congruence  $36x \equiv 15 \pmod{24}$ , if it exists.
4. Find the order of the ring  $M_2(\mathbb{Z}_2)$
5. Find a solution of the quadratic equation  $x^2 + 2x + 4 = 0$  in the ring  $\mathbb{Z}_6$
6. Find the number of zero divisors in the ring  $\mathbb{Z}_4$
7. Compute the product  $(12)(16)$  in  $\mathbb{Z}_{24}$

8. State whether true or false: " $\mathbb{Z}$  is a subfield of  $\mathbb{Q}$ "
9. Find all ideals of  $\mathbb{Z}_{12}$
10. Find the characteristic of the ring  $\mathbb{Z}_6 \times \mathbb{Z}_{15}$

(10 × 1 = 10 Marks)

SECTION – II

Answer **any eight** questions from this Section. Each carries 2 marks.

11. Let  $\phi: G \rightarrow G'$  be a group homomorphism of  $G$  onto  $G'$ . Prove that  $G'$  is abelian if  $G$  is abelian.
12. Find  $\ker \phi$  and  $\phi(3)$  for  $\phi: \mathbb{Z}_{10} \rightarrow \mathbb{Z}_{20}$  such that  $\phi(1) = 8$ .
13. Let  $\phi: G \rightarrow G'$  be a group homomorphism, show that if  $|G|$  is finite, then  $|\phi[G]|$  is finite and is a divisor of  $|G|$ .
14. Find the order of  $5 + \langle 4 \rangle$  in  $\mathbb{Z}_{12} / \langle 4 \rangle$ .
15. Show that  $A_n$  is a normal subgroup of  $S_n$  and compute  $S_n / A_n$ .
16. Prove that the factor group of a cyclic group is cyclic.
17. Compute the factor group  $(\mathbb{Z}_4 \times \mathbb{Z}_6) / \langle 0, 2 \rangle$ .
18. Let  $H$  be a normal subgroup of an abelian group  $G$ . Then show that  $G/H$  is abelian.
19. Describe all ring homomorphisms of  $\mathbb{Z}$  into  $\mathbb{Z}$ .
20. Find the remainder when  $3^{47}$  is divided by 23.
21. Describe all units in the ring  $\mathbb{Z} \times \mathbb{Z}$ .
22. Let  $R$  be a commutative ring with unity of characteristic 3. Compute and simplify  $(a+b)^6$  for  $a, b \in R$ .

(8 × 2 = 16 Marks)

### SECTION – III

Answer **any six** questions from this Section. Each carries **4** marks.

23. Prove that a group homomorphism  $\phi: G \rightarrow G'$  is a one-one map if and only if  $\ker \phi = \{e\}$
24. Let  $\phi: G \rightarrow H$  be a group homomorphism. Show that  $\phi[G]$  is abelian if and only if, for every  $x, y \in G$  we have  $xyx^{-1}y^{-1} \in \ker \phi$ .
25. Show that arbitrary intersection of normal subgroups of a group  $G$  is again a normal subgroup.
26. Show that the characteristic of an integral domain must be either zero or a prime  $p$ .
27. Find the last two digits in the decimal representation of  $3^{256}$ .
28. Show that for every integer  $n$ , the number  $n^{33} - n$  is divisible by 15.
29. Let  $d = \gcd(a, m)$ . Prove the congruence  $ax \equiv b \pmod{m}$  has a solution if and only if  $d \mid b$
30. Show that the group homomorphism  $\phi: G \rightarrow G'$  where  $|G|$  is prime must either be trivial or a one-one map
31. State and prove Fermat's Little Theorem.

**(6 × 4 = 24 Marks)**

### SECTION – IV

Answer **any two** questions from this Section. Each carries **15** marks.

32. Let  $R$  be a ring that contains at least two elements. Suppose for each non-zero  $a \in R$ , there exists a unique  $b \in R$  such that  $aba = a$ .
  - (a) Show that  $R$  has no divisors of zero.
  - (b) Show that  $bab = b$ .
  - (c) Show that  $R$  has unity.
  - (d) Show that  $R$  is a division ring.

33. (a) Show that all automorphisms of a group  $G$  form a group under function composition.
- (b) Show that the inner automorphisms of a group  $G$  form a normal subgroup of the group of all automorphisms of  $G$  under function composition.
34. State and prove fundamental theorem of Ring Homomorphism.
35. Prove that
- (a) Every field is an Integral Domain.
- (b) Every finite integral domain is a field.
- (c) If  $p$  is a prime, then  $\mathbb{Z}_p$  has no divisors of zero.

(2 × 15 = 30 Marks)