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Reg. No. : ....

Name : ....

Sixth Semester B.Sc. Degree Examination, April 2019
First Degree Programme under CBCSS
MATHEMATICS
Core Course – XII
MM 1644 : Abstract Algebra – II
(2014 Admn. Onwards)

Time : 3 Hours Max. Marks : 80

SECTION – I

All the first **10** questions are **compulsory**. **Each** carries **1** mark.

1. Find $\phi(25)$ for the homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}_7$ such that $\phi(1) = 4$.
2. How many homomorphisms are there of $\mathbb{Z}$ into $\mathbb{Z}$?
3. Find the order of the factor group $(\mathbb{Z}_4 \times \mathbb{Z}_2)/\langle(2, 1)\rangle$.
4. The trivial subgroup $N = \{0\}$ of $\mathbb{Z}$ is a normal subgroup. Compute $\mathbb{Z}/\{0\}$.
5. The image of a group of 6 elements under a homomorphism may have 12 elements. True or False.
6. Compute the product $(-3, 5)(2, -4)$ in the ring $\mathbb{Z}_4 \times \mathbb{Z}_{11}$.
7. Find all units in the ring $\mathbb{Z} \times \mathbb{Z}$.
8. Find the characteristic of the ring $\mathbb{Z}_3 \times \mathbb{Z}_3$.
9. Using Fermat's theorem, find the remainder of 3^{47} when it is divided by 23.
10. A ring homomorphism $\phi : R \to R'$ carries ideals of R into ideals of R'. True or False.

SECTION – II

Answer **any 8** questions from this Section. **Each** question carries **2** marks.

11. Show that a group homomorphism $\phi : G \to G'$ is a one-to-one map if and only if $Ker(\phi) = \{e\}$.

12. Let H be a normal subgroup of G. Then show that $\gamma : G \to G/H$ given by $\gamma(x) = xH$ is a homomorphism with kernel H.

13. Does there exist a nontrivial homomorphism $\phi : \mathbb{Z}_3 \to \mathbb{Z}$? If yes, give an example. If not, explain why that is so.

14. Show that any group homomorphism $\phi : G \to G'$ where $|G|$ is a prime must either be the trivial homomorphism or a one-to-one map.

15. Show that a factor group of a cyclic group is cyclic.

16. Let (R, +) be an abelian group. Show that (R, +, .) is a ring if we define ab = 0 for all a, b $\in$ R.

17. Are the fields $\mathbb{R}$ and $\mathbb{C}$ isomorphic ? Justify your answer.

18. In the ring $\mathbb{Z}_n$, show that the divisors of 0 are precisely those nonzero elements that are not relatively prime to n.

19. Show that 1 and p – 1 are the only elements of the field $\mathbb{Z}_p$ that are their own multiplicative inverse.

20. Let F be the ring of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ and having derivatives of all orders. Differentiation gives a map $\delta : F \to F$ where $\delta(f(x)) = f'(x)$. Is δ a homomorphism ? Why ?

21. Show that each homomorphism from a field to a ring is either one to one or maps everything onto 0.

22. Show that if R is a ring with unity and N is an ideal of R such that $N \neq R$, then R/N is a ring with unity.

SECTION – III

Answer **any 6** questions from this Section. **Each** question carries **4** marks.

23. Let $\phi : G \to G'$ be a group homomorphism. Show that if $|G|$ is finite, then $|\phi[G]|$ is finite and is a divisor of $|G|$.

24. Show that if a finite group G has exactly one subgroup H of a given order, then H is a normal subgroup of G.

25. Show that an intersection of normal subgroups of a group G is again a normal subgroup of G.

26. Show that if U is the collection of all units in a ring (R, +, .) with unity, then (U, .) is a group.

27. Show that every finite integral domain is a field.

28. Find all positive integers n such that $\mathbb{Z}_n$ contains a subring isomorphic to $\mathbb{Z}_2$.

29. Find all solutions of the congruence $155x \equiv 75 \pmod{65}$.

30. Let R be a commutative ring with unity of prime characteristic p. Show that the map $\phi_p : R \to R$ given by $\phi_p(a) = a^p$ is a homomorphism.

31. A ring R is a Boolean ring if $a^2 = a$ for all $a \in R$. Show that every Boolean ring is commutative.

SECTION – IV

Answer **any 2** questions from this Section. **Each** question carries **15** marks.

32. a) Prove or disprove : If d divides the order of G, then there must exist a subgroup H of G having order d. **10**

b) Let ϕ be a homomorphism of a group G into a group G'. If K' is a subgroup of G', then show that $\phi^{-1}[K']$ is a subgroup of G. **5**

33. a) Let $\phi : G \to G'$ be a homomorphism with kernel H and let $a \in G$. Prove the set $\{x \in G \mid \phi(x) = \phi(a)\} = Ha$. **5**

b) Let H be a normal subgroup of G. Show that the cosets of H form a group G/H under the binary operation (aH) (bH) = (ab)H. **5**

c) Show that if H and N are subgroups of a group G, and N is normal in G, then $H \cap N$ is normal in H. Show by an example that $H \cap N$ need not be normal in G. **5**

34. a) An element a of a ring R is idempotent if $a^2 = a$. Find all idempotents in the ring $\mathbb{Z}_6 \times \mathbb{Z}_{12}$. **5**

b) Show that the unity element in a subfield of a field must be the unity of the whole field. **5**

c) Solve the equation $x^2 - 5x + 6 = 0$ in $\mathbb{Z}_{12}$. **5**

35. a) Show that a division ring contains exactly two idempotent elements. **5**

b) Show that the characteristic of a subdomain of an integral domain D is equal to the characteristic of D. **5**

c) Show that $2^{11,213} - 1$ is not divisible by 11. **5**
