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**L – 1562**

**Reg. No. : ............................**

**Name : ..................................**

**Sixth Semester B.Sc. Degree Examination, March 2021**

**First Degree Programme under CBCSS**

**Mathematics**

**Core Course XI**

**MM 1643 : ABSTRACT ALGEBRA-RING THEORY**

**(2018 Admission Regular)**

Time : 3 Hours

Max. Marks : 80

SECTION – I

Answer **all** the first 10 questions. Each carries **1** mark.

1. Give an example of a non-commutative ring with unity.

2. Write a subring of $\mathbf{Z}_6$, the integers modulo 6.

3. Define the term "Zero divisors".

4. Why $\mathbf{Z}_{10}$, the integers modulo 10 is not an integral domain.

5. Find the characteristic of the integral domain $\mathbf{Z}_{19}$, the integers modulo 19.

6. List the elements in $2\mathbb{Z}/6\mathbb{Z}$.

7. Show that the correspondence $x \mapsto 3x$ from $\mathbb{Z}_4$ to $\mathbb{Z}_{12}$ does not preserve multiplication.

8. Give an example of an integral domain which is not a unique factorization domain.

P.T.O.

9. True or False : "The ring of Gaussian integers a unique factorization domain".

10. In the ring of integers, find a positive integer $a$ such that $\langle a \rangle = \langle 6 \rangle + \langle 8 \rangle$.

**(10 × 1 = 10 Marks)**

SECTION – II

Answer **any eight** questions among the questions 11 to 26. They carry **2** marks each.

11. Let $\phi : \mathbf{R}[x] \mapsto \mathbf{C}$ be a homomorphism with the property that $\phi(x) = \phi(i)$. Evaluate $\phi(x^2 + 1)$.

12. Show that the polynomial $2x + 1$ in $\mathbb{Z}_4[x]$ has a multiplicative inverse in $\mathbb{Z}_4[x]$.

13. Show that the polynomial $2x^2 + 4$ is not reducible over $\mathbb{Q}$ but reducible over $\mathbb{Z}$.

14. Suppose that $R$ is an integral domain in which $20 * 1 = 0$ and $12 * 1 = 0$. What is the characteristic of $R$?

15. Let $D$ be a Euclidean domain with measure $d$. Show that if $a$ and $b$ are associates in $D$, then $d(a) = d(b)$.

16. Show that $\mathbb{Z}[\sqrt{-6}]$ is not a unique factorization domain.

17. If $a$ and $b$ belong to $\mathbb{Z}[\sqrt{d}]$, where $d$ is not divisible by the square of a prime and $ab$ is a unit, power that $a$ and $b$ are units.

18. Give an example of ring elements $a$ and $b$ with the properties that $ab = 0$ but $ba \neq 0$.

19. Prove that "Let $a$, $b$ and $c$ belong to an integral domain. If $a \neq 0$ and $ab = ac$, then $b = c$".

20. Prove that the only idempotents in an integral domain are 0 and 1.

21. Consider the equation $x^2 - 5x + 6 = 0$. Find all solutions of this equation in $\mathbb{Z}_8$.

22. Find a subring of $\mathbb{Z} \oplus \mathbb{Z}$ that is not an ideal of $\mathbb{Z} \oplus \mathbb{Z}$.

23. Draw the lattice diagram of ideals of $\mathbb{Z}_{36}$.

24. Give an example of a commutative ring that has a maximal ideal that is not a prime ideal.

25. Show that the mapping $a+ib$ to $a-ib$ is a ring isomorphism from the complex numbers onto the complex numbers.

26. Give an example of a ring with unity 1 that has a subring with unity $1'$ such that $1' \neq 1$.

**(8 × 2 = 16 Marks)**

SECTION – III

Answer **any six** questions among the questions 27 to 38. They carry **4** marks each.

27. If $R$ is a ring, then for any $a, b \in R$, show that $a(-b) = (-a)b = -(ab)$.

28. Show that "If $p$ is a prime, then $\mathbb{Z}_p$ is a field".

29. Let $F$ be a field of order $2^n$. Prove that characteristic of $F = 2$.

30. Let $p$ be a prime. Show that in the ring $\mathbb{Z}_p$ you have $(a+b)^p = a^p + b^p$.

    for every $a, b \in \mathbb{Z}_p$.

31. Let $R$ be a ring and let $I$ be an ideal of $R$. Prove that the factor ring $R/I$ is commutative if and only if $rs - sr \in I$ for all $r$ and $s$ in $R$.

32. Find all ring homomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{30}$.

33. Show that "If $D$ is an integral domain, then $D[x]$ is an integral domain".

34. Find the quotient and remainder upon dividing $f(x) = 3x^4 + x^3 + 2x^2 + 1$ by $g(x) = x^2 + 4x + 2$.

35. By stating necessary theorem show that the polynomial $3x^5 + 15x^4 - 20x^3 + 10x + 20$ is irreducible over $\mathbb{Q}$, the set of rational numbers.

36. Prove that "In an integral domain, every prime is an irreducible".

37. Let $F$ be a field and let $a$ be a non zero element of $F$. If $f(x+a)$ is irreducible over $F$, prove that $f(x)$ is irreducible over $F$.

38. Let $D$ be a Euclidean domain with measure $d$. Prove that $u$ is a unit in $D$ if and only if $d(u) = d(1)$.

**(6 × 4 = 24 Marks)**

## SECTION – IV

Answer **any two** questions among the questions 39 to 44. They carry **15** marks each.

39. Prove that "Let $R$ be a commutative ring with unity and let $A$ be an ideal of $R$. Then $R/A$ is an integral domain if and only if $A$ is prime".

40. (a) Let $a$ and $b$ be idempotents in a commutative ring. Show that each of the following is also an idemptotent :

    (i) $ab$

    (ii) $a - ab$

    (iii) $a + b - ab$

    (iv) $a + b - 2ab$.

    (b) Show that a unit of a ring divides every element of the ring.

41. (a) Prove that "Let $\phi$ be a ring homomorphism from a ring $R$ to a ring $S$. Then $Ker\phi = \{r \in R; \phi(r) = 0\}$ is an ideal of $R$".

    (b) Show that $\phi : \mathbb{Z}_4 \mapsto \mathbb{Z}_{10}$ by $\phi(x) = 5x$ is a ring homomorphism.

42. Prove that "A polynomial of degree $n$ over a field has at most $n$ zeros, counting multiplicity".

43. Prove that "Let $f(x) \in \mathbb{Z}[x]$. If $f(x)$ is reducible over $\mathbb{Q}$, then it is reducible over $\mathbb{Z}$".

44. In $\mathbb{Z}[i]$, show that 3 is irreducible but 2 and 5 are not.

**(2 × 15 = 30 Marks)**

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