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Reg. No. : ....

Name : ....

Sixth Semester B.Sc. Degree Examination, March 2021

First Degree Programme under CBCSS

Mathematics

Core Course IX

MM 1641 : REAL ANALYSIS - II

(2015-2017 Admission)

Time : 3 Hours Max. Marks : 80

SECTION – I

All the first **ten** questions are compulsory. Each question carries **1** mark.

1. State $\in-\delta$ definition of continuity of a function.

2. State Discontinuity Criterion.

3. Determine the points of continuity of the function defined by $f(x) = [[\sin x]]$, $x \in \mathbb{R}$ where $[[x]]$ denotes the greatest integer less than or equal to x.

4. Give an example of a function $f:[0, 1] \rightarrow \mathbb{R}$ that is discontinuous at every point of $[0, 1]$ but $|f|$ is continuous on $[0, 1]$.

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5. State Boundedness theorem.

6. Give an example of a function which is monotone but not continuous.

7. Find the points of relative extrema of the function $f(x) = \left| x^2 - 1 \right|$ for $-4 \le x \le 4$.

8. State Cauchy Mean value theorem.

9. Evaluate $\lim_{x \to \infty} \frac{x^2}{e^x}$.

10. Define Riemann integrable function.

(10 × 1 = 10 Marks)

SECTION – II

Answer **any eight** questions from this section. Each question carries **2** marks.

11. Discuss the continuity of the function $F : \mathbb{R} \to \mathbb{R}$ at $x = 0$, defined by

$$F(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \end{cases}$$

12. Show that rational function is continuous at every real number for which it is defined.

13. Give an example of functions f and g that are both discontinuous at a point c in $\mathbb{R}$ such that the sum $f + g$ is continuous at c.

14. Show that if $f: I \to \mathbb{R}$ has a derivative a $c \in I$ then f is continuous at c.

15. Show that the function $f(x) = x^{1/3}$, $x \in \mathbb{R}$ is not differentiable at $x = 0$.

16. If $a > 0, b > 0$ and α be any real number with $0 < \alpha < 1$ then prove that $a^{\alpha} b^{1-\alpha} \leq \alpha\, a + (1-\alpha)^{b}$ where equality holds if and only if $a = b$.

17. Let f and g be defined on [a, b], let $f(a) = g(a) = 0$, and let $g(x) \neq 0$ for $a < x < b$. If f and g are differentiable at a and if $g'(a) \neq 0$, then prove that limit of $\frac{f}{g}$ at a exists and is equal to $\frac{f'(a)}{g'(a)}$.

18. Evaluate $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x}$.

19. Let $f(x) = x$ for x in [0, 1]. Prove that $f \in \mathcal{R}$ [0, 1] and $\int_{0}^{1} f(x)\,dx = \frac{1}{2}$.

20. Using substitution theorem, evaluate $\int_{1}^{4} \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt$.

21. Prove that the Dirichlet function $f: [0, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 0 \text{ if } x \text{ is irrational} \\ 1 \text{ if } x \text{ is rational} \end{cases}$ is not Riemann integrable.

22. Find $\int_{-10}^{10} \operatorname{sgn}(x)\,dx$.

(8 × 2 = 16 Marks)

SECTION – III

Answer **any six** questions from this section. Each question carries **4** marks.

23. Let $A = \{x \in \mathbb{R} : x > 0\}$. Define $h : A \to \mathbb{R}$ by

$$h(x) = \begin{cases} 0, \text{ if } x \text{ is irrational} \\ \frac{1}{n}, \text{ if } x = \frac{m}{n} \text{ is rational} \end{cases}$$

Where m and n are natural numbers having no common factors except 1. Show that h is continuous at every irrational number in A, and is discontinuous at every rational number in A.

24. Let $A, B \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be continuous on A, and let $g : B \to \mathbb{R}$ be continuous on B. If $f(A) \subseteq B$, prove that the composite function $g \circ f : A \to \mathbb{R}$ is continuous on A.

25. Prove that if f and g are two functions differentiable at c then their product fg is differentiable at c.

26. Given that the function $f(x) = x^n$, $x \geq 0$, $n \in \mathbb{N}$, has an inverse g on $(0, \infty)$, find $g'(y)$ for $y > 0$.

27. State and prove Mean value theorem.

28. Let $f : I \to \mathbb{R}$ be differentiable on the interval I. Prove that f is increasing on I if and only if $f'(x) \geq 0$ for all $x \in I$.

29. State and prove Darboux's theorem.

30. If $f \in \mathcal{R}[a, b]$, then prove that the value of the integral is uniquely determined.

31. Prove that if $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then $f \in \mathcal{R}[a, b]$.

(6 × 4 = 24 Marks)

SECTION – IV

Answer **any two** questions from this section. Each question carries **15** marks.

32. (a) State and prove Maximum-Minimum theorem. **8**

(b) State and prove chain rule of differentiation. **7**

33. (a) Let I be an interval and let $f : I \to \mathbb{R}$ be strictly monotone on I. Let $J = f(I)$ and let $g : J \to \mathbb{R}$ be the function inverse to f. If f is differentiable on I and $f'(x) \neq 0$ for $x \in I$, then prove that g is differentiable on J and $g' = \frac{1}{f' \circ g}$. **8**

(b) State and prove Caratheodory's theorem. **7**

34. (a) State and prove L'Hospital's Rule-I. **10**

(b) If f and g are in $\mathcal{R}[a, b]$ and $f(x) \leq g(x)$ for all x in $[a, b]$ then show that $\int_a^b f(x)dx \leq \int_a^b g(x)dx$. **5**

35. (a) If $f \in \mathcal{R}[a, b]$, prove that f is bounded on $[a, b]$. 7

(b) State and prove Fundamental theorem of Calculus (Second Form). 8

(2 × 15 = 30 Marks)