Third Semester

Mathematics

MODULE - 1

1 mark

1) Express $x = BA_{twelve}$ in base ten.

2) Multiply $1011_{two}$ and $101_{two}$.

3) Let $f(n)$ denote the number of positive integers $\leq n$ and relatively prime to it. Find $f(24)$.

4) State the division algorithm.

5) State the pigeonhole principle.

6) Find five consecutive integers that are composites.

7) State the prime number theorem.

8) If $p$ is a prime and if $p|ab$, then prove that $p|a$ or $p|b$.

9) Express $3ABC_{sixteen}$ in base ten.

2 marks

1) Find the base $b$ if $1001_b = 126$.

2) Express $(28,12)$ as a linear combination of 28 and 12.
3) Prove that every integer $n \geq 2$ has a prime factor.
4) Show that $n^3 - n$ is divisible by 2.



1

5) Express 3014 in base eight.
6) Using recursion, evaluate (18, 30, 60, 75, 132).
7) Let b be an integer $\geq 2$. Suppose b+1 integers are randomly selected. Prove that the difference of two of them is divisible by b.
8) State the Inclusion Exclusion principle.
9) If $a|c$ and $b|c$, and $(a,b)=1$, show that $ab|c$.
10) If p is a prime and $p|ab$, show that $p|a$ or $p|b$.
11) Find the primes such that their digits in the decimal values alternate between 0's and 1's, beginning with the ending in 1.
12) If $a|c$ and $b|c$, can we say that $ab|c$? Justify your answer.
13) Find the number of positive integers $\leq 3000$ and divisible by 3, 5 or 7.
14) Show that 111 cannot be a square in any base.
15) Verify whether the LEDs $12x+18y=30$ & $6x+8y=25$ are solvable.

4 marks

1) Prove that there are infinitely many primes.
2) Let a and b be positive integers. Then prove that $[a,b] = \frac{ab}{(a,b)}$. Using the canonical decomposition of 18 and 24, find their LCM.
3) State Inclusion-Exclusion Principle. Find the number of positive integers in the range 1976 through 3776 that are divisible by 13 or 15.
4) Show that the number of leap years after 1600 and not exceeding a given year y is given by $l = \lfloor y/4 \rfloor - \lfloor y/100 \rfloor + \lfloor y/400 \rfloor - 388$.
5) Show that "If p and $p^2+2$ are primes, then $p^3+2$ is also a prime."





2

6) A six-digit positive integer is cut up in the middle into two three-digit numbers. If the square of their sum yields the original number, find the number.

7) Let a and b be any positive integers. Show that the number of positive integers $\leq a$ and divisible by b is $\lfloor a/b \rfloor$.

8) Find the primes such that the digits in their decimal values alternate between 0's and 1's, beginning with 1 and ending in 1.

9) Let $f_i$ denote the i-th Fermat number. Show that $f_0 f_1 \cdots f_{n-1} = f_n - 2$, where $n \geq 1$.

10) Show that there are infinitely many primes of the form $4n+3$.

11) Find the number of trailing zeros in 234!.

12) Let be be an integer $\geq 2$. Suppose b+1 integers are randomly selected. Prove that the difference of two of them is divisible by b.

13) Let a and b be positive integers. Derive a relationship between (a,b) and [a,b]. Also verify it for the integers 18 and 24.

15 marks

1) (a) Let a be any integer and b a positive integer. Then there exist unique integers q and r such that $a = bq + r$ where $0 \leq r < b$.

(b) Prove that there are atleast $3\lfloor \frac{n}{2} \rfloor$ primes in the range n through n!, where $n \geq 4$.

(c) Prove that every integer $n \geq 2$ either is a prime or can be expressed as a product of primes. The





3

factorization into primes is unique except for the order of the factors.

2) (a) Find the number of positive integers $\leq 3000$ and divisible by 3, 5 or 7.
(b) Every positive integer n can be written as $n = 2^a 5^b c$, where c is not divisible by 2 or 5.
(c) Find the canonical decomposition of 2520.

3) (a) State and prove the fundamental theorem of arithmetic.
(b) Show that the linear Diophantine equation (LDE) $ax+by=c$ is solvable if and only if $d|c$, where $d=(a,b)$. Also show that, if $x_0, y_0$ is a particular solution of the LDE, then all its solutions are given by $x = x_0 + \left(\frac{b}{d}\right)t$, $y = y_0 - \left(\frac{a}{d}\right)t$, where t is an arbitrary integer.

4) (a) Let $\alpha = \frac{1+\sqrt{5}}{2}$. Show that $\alpha^{n-2} < F_n < \alpha^{n-1}$; where $n \geq 3$ and $F_n$ denotes the n-th Fibonacci number.
(b) Show that the number of divisions needed to compute (a, b) by the euclidean algorithm is at most five times the number of decimal digits in b, where $a \geq b \geq 2$.

5) (a) A six-digit positive integer is cut up in the middle into two three-digit numbers. If the square of their sum yields the original, find the number.
(b) Solve the LDE $1076x + 2076y = 3076$ by Euler's method.





4

6) (a) State and prove the fundamental theorem of arithmetic.

(b) Explain the Euclidean algorithm and evaluate (4076, 1024).

1 mark

10) If $(x_0, y_0)$ is a particular solution of the Linear Diophantine Equation $ax+by=c$, write its general solution.





5

(1) Sketch the contour plot of $f(x,y) = 4x^2+y^2$ using level curves of height 2. (2 marks).

(2) Let $f(x,y) = \sqrt{3x+2y}$. Find the slope of the surface $z = f(x,y)$ in $x$-direction at the point $(4, 2)$.

(3) Compute differential $dw$ of the function $w = x^3y^2z$.

(4) Find $\frac{\partial z}{\partial u}$ for $z = 8x^y - 2x + 3y$, $x = uv$, $y = u-v$.

(5) Let $f(x,y) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2} & , \text{ if } (x,y) \neq (0,0) \\ 1 & , \text{ if } (x,y) = (0,0) \end{cases}$

Show that $f$ is continuous at $(0,0)$.

(6) Find the equation of the tangent plane to the ellipsoid $x^2+4y^2+z^2=18$ at the point $(1,2,1)$.

(7) Let $f(x,y) = x^2e^y$. Find the maximum value of a directional derivative at $(-2,0)$ and find the unit vector in the direction in which the maximum value occurs.

(8) Using chain rule find $\frac{dz}{dt}$ if $z = x^2y$, $x = t^2$, $y = t^3$.

(9) Show that $f(x,y) = x^2+y^2$ is differentiable at $(0,0)$.

(10) Find $f_x$ and $f_y$ for $f(x,y) = 2x^3y^2 + 2y + 4x$.

(11) Evaluate $\lim\limits_{(x,y)\to(0,0)} \frac{-xy}{x^2+y^2}$ along the parabola $y = x^2$.

(12) Suppose $w = xy + yz$, $y = \sin x$, $z = e^x$. Use chain rule to find $\frac{dw}{dx}$.

(13) Find the largest region on which $f(x,y,z) = 3x^2e^{yz}\cos(xyz)$ is continuous.

(14) Given $f(x,y) = x^3y^5 - 2x^2y + x$. Find $f_{xxy}$ and $f_{yxy}$.

(15) Find an equation for the tangent plane to the surface $x^2+y^2+z^2=25$ at the point $P(-3,0,4)$.

(16) Find $\lim_{(x,y)\to(0,0)} \tan^{-1}\left(\frac{x^2+1}{x^2+(y-1)^2}\right)$.

(17) Find $\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$.

(18) Verify: If $F(x,y,z) = 2z^3 - 3(x^2+y^2)z$, then $F_{xx}+F_{yy}+F_{zz}=0$.

(19) State the second partial test

(20) Compute 2nd order partial derivatives of $f(x,y) = x^2y^3 + x^4y$.

(1) Show that $\lim_{(x,y)\to(0,0)} \frac{x+y}{2x^2+y^2}$ does not exist by considering the limit as $(x,y)\to(0,0)$ along the coordinate axes. (4 marks)

(2) Find the directional derivative of $f(x,y) = \sinh x \cosh x$ at $(0,0)$ in the direction of a vector making counter clockwise angle $\theta = \pi$ with the positive $x$-axis.

(3) Find the parametric equations of the tangent line to the curve of intersection of the paraboloid $z = x^2+y^2$ and the ellipsoid $3x^2+2y^2+z^2=9$ at the point $(1,1,2)$.

(4) Find the directional derivative of $f(x,y) = e^{xy}$ at $(-2,0)$ in the direction of the unit vector making an angle $\pi/3$ with the +ve $x$-axis.

(5) Suppose $w = \sqrt{x^2+y^2+z^2}$, $x = \cos\theta$, $y = \sin\theta$, $z = \tan\theta$. Use chain rule to find $\frac{dw}{d\theta}$ at $\theta = \pi/4$.

(6) Confirm that the mixed partial derivatives of $f(x,y) = 4x^2 - 8xy^4 + 7y^4 - 3$ are equal.

(7) Locate all relative maxima, relative minima, and saddle points if any for the function $f(x,y) = y^2+xy+3y+2x+3$.

(8) Find the point on the plane $x+2y+3z = 13$ closest to the point $(1,1,1)$.

(9) Find the dimensions of the closed right circular cylinder can of smallest surface area whose volume is $16\pi$ cm$^3$.

(10) Use the method of Lagrange's multiplier to find the dimensions of a rectangle with perimeter $p$ and maximum area.

(11) For the function $f(x,y) = \dfrac{xy}{x^2+y^2}$ estimate the limit of $f(x,y)$ as $(x,y) \to (0,0)$ along

(a) x-axis

(b) y-axis

(c) the line $y = x$.

(d) parabola $y = x^2$.

(12) Given that $z = e^{xy}$, $x = 2u+v$, $y = \dfrac{u}{v}$. Compute $\dfrac{\partial z}{\partial u}$, $\dfrac{\partial z}{\partial v}$.

① (15 marks)

ⓐ If $f$ and $g$ are differentiable function of $x$ and $y$ then prove that $\nabla(fg) = f\nabla g + g\nabla f$.

ⓑ Find the slope of the sphere $x^2+y^2+z^2=1$ in the $y$-direction at the point $(2/3, 1/3, 2/3)$.

ⓒ Find the parametric equation of the tangent line to the curve of intersection of the paraboloid $z = x^2+y^2$ and the ellipsoid $x^2+4y^2+z^2=9$ at the point $(1,1,2)$.

② ⓐ Define relative maxima for a function of 2 variables at a point $(x_0, y_0)$.

ⓑ What is critical point of a function of 2 variables.

ⓒ Give an example of a bounded set in $xy$-plane.

ⓓ Find the dimensions of a rectangular box of maximum volume that can be inscribed in a sphere of radius $a$.

③ ⓐ The length, width, and height of a rectangular box are measured with an error of at most 5%. Use a total differential to approximate the maximum percentage error that results if these quantities are used to calculate the diagonal of the box.

ⓑ Let $L(x, y)$ denote the local linear approximation to $f(x,y) = \sqrt{x^2+y^2}$ at the point $(3,4)$. Compare the error in approximating $f(3.04, 3.98)$ by $L(3.04, 3.98)$ with the distance between the points $(3,4)$ and $(3.04, 3.98)$.

④ ① Find the absolute maximum and minimum values for $f(x,y) = 3xy - 6x - 3y + 7$ on the closed triangular region $R$

with vertices $(0,0,)$ $(3,0)$ and $(0,5)$.

(2) Consider ellipsoid $x^2+4y^2+z^2=18$.

(i) Find the equation of the tangent plane to the ellipsoid at the point $(1,2,1)$.

(ii) Find parametric equation of the line that is normal to the ellipsoid at the point $(1,2,1)$.

(iii) Find the acute angle that the tangent plane at the point $(1,2,1)$ makes with the XY-plane.

(5) (a) Locate relative extrema and saddle points of $f(x,y) = 4xy - x^4 - y^4$.

(b) Find the points on the sphere $x^2+y^2+z^2=36$ that are closest to and farthest from the point $(1,2,2)$.

(6) Use Lagranges' multiplier to determine the dimensions of a rectangular box, open at the top, having volume of $32\ ft^3$ and requiring the least amount of material for its construction.